

CHERN–MATHER CLASSES OF TORIC VARIETIES

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ABSTRACT. The purpose of this short note is to prove a formula for the Chern–Mather classes of a toric variety in terms of its orbits and the local Euler obstructions at general points of each orbit (Theorem 2). We use the general definition of the Chern–Schwartz–MacPherson classes (see [7]) and their special expression in case of a toric variety (see [2]). As a corollary, we obtain a formula by Matsui–Takeuchi [8, Corollary 1.6]. Alternatively, one could deduce the formula of Theorem 2 from the Matsui–Takeuchi formula, by using our general result [11, Théorème 3] for the degree of the polar varieties in terms of the Chern–Mather classes.

We first recall the definition of the Chern–Mather class $c^M(X)$ of an n -dimensional variety X . Let $\tilde{X} \subseteq \text{Grass}_n(\Omega_X^1)$ denote the Nash transform of X , i.e., \tilde{X} is the closure of the graph of the rational section of $\text{Grass}_n(\Omega_X^1)$ given by the locally free rank n sheaf $\Omega_X^1|_{X_{\text{sm}}}$. We set $c^M(X) := \nu_*(c(\Omega^\vee) \cap [X])$, where Ω is the tautological sheaf on $\text{Grass}_n(\Omega_X^1)$ and $\nu: \tilde{X} \rightarrow X$.

The *polar loci* of an n -dimensional projective variety $X \subset \mathbb{P}^N$ are defined as follows: Let $L_k \subset \mathbb{P}^N$ be a linear subspace of codimension $n - k + 2$. The polar locus of X with respect to L_k is

$$M_k := \overline{\{x \in X_{\text{sm}} \mid \dim(T_{X,x} \cap L_k) \geq k - 1\}},$$

where $T_{X,x}$ denotes the projective tangent space to X at the (smooth) point x . (For other interpretations of M_k , see e.g. [11].) The rational equivalence classes $[M_k]$ are independent of L_k , for general L_k , and (in 1978) we showed the following:

Theorem 1. [11, Théorème 3] *The polar classes of X are given by*

$$[M_k] = \sum_{i=0}^k (-1)^i \binom{n-i+1}{n-k+1} h^{k-i} \cap c_i^M(X), \quad (1)$$

and, reciprocally, the Chern–Mather classes of X are given by

$$c_k^M(X) = \sum_{i=0}^k (-1)^i \binom{n-i+1}{n-k+1} h^{k-i} \cap [M_i], \quad (2)$$

where h is the class of a hyperplane.

Recall (see [7]) that the Chern–Schwartz–MacPherson class of X is defined by

$$c^{\text{MP}}(X) := c^M \circ T^{-1}(\mathbf{1}_X).$$

Here we define $c^M: Z(X) \rightarrow A(X)$ by $c^M(\sum n_i V_i) = \sum n_i c^M(V_i)$, and T is the isomorphism from the group of cycles $Z(X)$ to the group of constructible functions on X , given by

$$T(V)(x) = \text{Eu}_V(x),$$

where Eu_V denotes the constructible function whose value at a point $x \in X$ is equal to the local Euler obstruction of V at x (hence is 0 if $x \notin V$). Note that $\text{Eu}_V(x) = 1$ if $x \in V$ is a smooth point, but that the converse is false (this was first observed in [11, Example, pp. 28–29]). The Chern–Schwartz–MacPherson classes are invariant under homeomorphisms, whereas the Chern–Mather classes are invariant under generic linear projections (since the polar varieties are [10, Thm. 4.1, p. 269]).

In what follows we shall consider *toric* varieties, defined as follows. Let $A \subset \mathbb{Z}^n$ be a set of $N + 1$ points such that the polytope $P := \text{Conv}(A) \subseteq \mathbb{R}^n$ has dimension n . Let $X := X_A \subset \mathbb{P}^N$ denote the corresponding toric variety. Let $\{X_\alpha\}_\alpha$ denote the orbits of the torus action on X . The classes $[\overline{X}_\alpha]$ generate the Chow ring $A(X)$ [5, 5.1, Prop., p. 96]. Moreover, \overline{X}_α is the toric variety corresponding to the lattice points $A_\alpha := A \cap F_\alpha$, where F_α is the face of P corresponding to the orbit X_α . It was shown in [2, Théorème] that the Chern–Schwartz–MacPherson class of X is given by

$$c^{\text{MP}}(X) = \sum_{\alpha} [\overline{X}_\alpha]. \quad (3)$$

The purpose of this note is to prove Theorem 2 below, using (3). As a corollary we obtain formulas for the ranks (degrees of the polar varieties) of X , in particular the formula of [8, Corollary 1.6], hence we have an alternative proof of this result. Observe that if we instead *assume* [8, Corollary 1.6], then we can deduce Theorem 2 by using [11, Théorème 3].

Theorem 2. *The Chern–Mather class of the toric variety X is equal to*

$$c^{\text{M}}(X) = \sum_{\alpha} \text{Eu}_X(X_\alpha) [\overline{X}_\alpha],$$

where the sum is taken over all orbits X_α of the torus action on X , and where $\text{Eu}_X(X_\alpha)$ denotes the value of the local Euler obstruction of X at a point in the orbit X_α .

Proof. Write $T^{-1}(\mathbf{1}_X) = X + \sum_{\alpha} a_{\alpha} \overline{X}_{\alpha}$ for some $a_{\alpha} \in \mathbb{Z}$, so that $\mathbf{1}_X = T(X) + \sum_{\alpha} a_{\alpha} T(\overline{X}_{\alpha})$, where the sums are over α such that $\overline{X}_{\alpha} \neq X$. Then for $x \in X_{\beta}$, we get

$$1 = \text{Eu}_X(X_{\beta}) + \sum_{\alpha \succ \beta} a_{\alpha} \text{Eu}_{\overline{X}_{\alpha}}(X_{\beta}), \quad (4)$$

where the sum is over all orbits X_{α} such that $X \neq \overline{X}_{\alpha} \supset X_{\beta}$. We also have

$$c^{\text{MP}}(X) = c^{\text{M}} \circ T^{-1}(\mathbf{1}_X) = c^{\text{M}}(X) + \sum_{\alpha} a_{\alpha} c^{\text{M}}(\overline{X}_{\alpha}).$$

Using (3), this gives

$$c^{\text{M}}(X) = [X] + \sum_{\alpha} [\overline{X}_{\alpha}] - \sum_{\alpha} a_{\alpha} c^{\text{M}}(\overline{X}_{\alpha}), \quad (5)$$

where again the sum is over all α such that $\overline{X}_{\alpha} \neq X$.

We shall use induction on the dimension of X . If $\dim X = 1$, then there are two 0-dimensional orbits x_1 and x_2 . Thus (4) gives $1 = \text{Eu}_X(x_i) + a_i$, for $i = 1, 2$, so that $a_i = 1 - \text{Eu}_X(x_i)$. Hence (5) gives

$$c_1^{\text{M}}(X) = \sum [x_i] - \sum (1 - \text{Eu}_X(x_i)) [x_i] = \sum \text{Eu}_X(x_i) [x_i],$$

which is what we wanted to show.

Assume now that the theorem holds for toric varieties of dimension $< \dim X$. Then for each $\overline{X}_\alpha \neq X$ we can write

$$c^M(\overline{X}_\alpha) = \sum_{\beta \prec \alpha} \text{Eu}_{\overline{X}_\alpha}(X_\beta)[\overline{X}_\beta],$$

where the sum is over all β such that $X_\beta \subset \overline{X}_\alpha$. From (5) we get

$$c^M(X) = [X] + \sum_{\alpha} [\overline{X}_\alpha] - \sum_{\alpha} a_{\alpha} \sum_{\beta \prec \alpha} \text{Eu}_{\overline{X}_\alpha}(X_\beta)[\overline{X}_\beta].$$

Rewriting the last double sum as $\sum_{\beta} (\sum_{\alpha} a_{\alpha} \text{Eu}_{\overline{X}_\alpha}(X_\beta))[\overline{X}_\beta]$ and applying (4) gives the formula of the theorem. \square

Let $\mu_k := \deg M_k$ denote the degrees of the polar varieties of X . Applying the equality (1) of Theorem 1 we obtain:

Theorem 3. *The degrees of the polar varieties of the toric variety X are given by*

$$\mu_k = \sum_{i=0}^k (-1)^i \binom{n-i+1}{n-k+1} \sum_{\alpha} \text{Eu}_X(X_{\alpha}) \text{Vol}_{\mathbb{Z}}(F_{\alpha}),$$

where the second sum is over all α such that X_{α} has codimension i in X , and $\text{Vol}_{\mathbb{Z}}(F_{\alpha})$ denotes the lattice volume of the face F_{α} of P corresponding to the orbit X_{α} .

Corollary 4 (Matsui–Takeuchi [8, Corollary 1.6]). *Assume the dual variety of $X \subset \mathbb{P}^N$ is a hypersurface. Then its degree is given by*

$$\deg X^{\vee} = \sum_{F_{\alpha} \leq P} (-1)^{\text{codim } F_{\alpha}} (\dim F_{\alpha} + 1) \text{Eu}_X(X_{\alpha}) \text{Vol}_{\mathbb{Z}}(F_{\alpha}),$$

where X_{α} denotes the orbit in X corresponding to the face F_{α} of P .

Proof. In this case the degree of the dual variety is equal to μ_n . \square

Examples. The toric varieties we consider need not be normal, in particular the set of lattice points A need not be equal to the set of lattice points in $P = \text{Conv}(A)$. Note that we can view X_A as a (toric) linear projection of X_P . When this projection is “generic”, the Chern–Mather classes of X_A are just the pushdowns of the Chern–Mather classes of X_P [11, Corollaire, p. 20]. Here are two simple examples.

1) Let $A = \{(0,0), (0,1), (1,1), (2,0)\}$. Then $X_A \subset \mathbb{P}^3$ is a cubic surface with a double line with two pinch points, and it is the projection of a rational normal surface of type (1,2). The closure of the orbit of X_A corresponding to the line segment $[(0,0), (2,0)]$ has normalized lattice volume 1 and local Euler obstruction 2. The three other 1-dimensional orbits are smooth and have lattice volume 1. Moreover, as shown in [11, p. 29], the local Euler obstruction at a pinch point is 1. Hence we get

$$\deg X_A^{\vee} = 3 \cdot 3 - 2(2 \cdot 1 + 3) + 4 = 3.$$

Since any toric hypersurface X_A , where A is not a pyramid, is selfdual [3], this is of course no surprise. Note that we also get $\deg X_P^{\vee} = 3$.

2) Let $A = \{(0,0), (1,1), (0,2), (3,0)\}$. Then $X_A \subset \mathbb{P}^3$ is a (non-generic) toric linear projection of the weighted projective space $X_P = \mathbb{P}(1,2,3) \subset \mathbb{P}^6$. In this case

$\deg X_A^\vee = 6$, whereas $\deg X_P^\vee = 7$. Note that for X_A , the local Euler obstructions at the 1-dimensional orbits are 1, 2, and 3, whereas all the three 0-dimensional orbits have local Euler obstruction 0. (For more examples of explicit computations of the local Euler obstruction for toric varieties, especially in the case of weighted projective spaces, see [9].)

Remark. There has recently been a renewed interest in Chern–Mather classes and polar varieties, in particular related to the concept of Euclidean distance degree. This includes other types of polar varieties (see the survey [12] and the references given there). For a cycle theoretic approach, see [1]; for applications, see [4] for the general case and [6] for the toric case.

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